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Variants of free semigroups and free monoids

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Abstract. For any element x of a semigroup S, a variant of S is a semigroup on S with multiplication $*_x$ defined by $a *_x b = axb$. Automorphism groups of variants of the free monoid and isomorphism criteria for variants of free semigroups and free monoids are described.

Introduction

Let S be a semigroup and $x \in S$. Define a new product $*_x$ on S by putting

$$\forall a, b \in S : a *_x b = axb.$$

It is clear that $(S; *_x)$ is a semigroup. Operation $*_x$ is called a *deformed multiplication* and $(S; *_x)$ is called a *variant* of S. Denote $(S; *_x)$ by S_x.

Variants of abstract semigroups were first studied by J. Hickey [4]. Variants of regular semigroups were considered in [5], [6]. Variants of transformation semigroups also had been studied by many authors. The isomorphism criteria for variants of transformation semigroups and partial transformation semigroups were presented in [9–11]. Isolated semigroups of symmetric inverse and transformation semigroups with deformed multiplication were described in [9], [13].

Variants of a semigroup are naturally connected with interassociates of a semigroup. Let $(S; \cdot)$ and $(S; \circ)$ be semigroups defined on the same set. $(S; \circ)$ is called an *interassociate* of $(S; \cdot)$ if the next identities of an interassociativity are satisfied:

$$a \cdot (b \circ c) = (a \cdot b) \circ c,$$

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$$a \circ (b \cdot c) = (a \circ b) \cdot c.$$

Obviously, each variant of a semigroup defines an interassociate of it. It is also easy to show that every interassociate of a monoid is a variant thereof.

In [14], Zupnik introduced the term *interassociativity*. All interassociates of monogenic semigroups were described in [3]. In [1] Givens, Linton, Rosin and Dishman found isomorphism criteria for the free commutative semigroup on n generators. They have also shown that every interassociate of the finitely generated free commutative semigroup coincides with it or can be written as its variant, i.e. those isomorphism criteria are true for a variants of considered semigroups. In [2] similar results were obtained for the infinitely generated free commutative semigroup.

In this paper we describe isomorphism criteria for variants of free semigroups and free monoids. For the variants of free monoids we also describe automorphism groups and all isomorphisms between these objects.

1. Deformed power of words

For a nonempty set X ($|X| > 1$), we denote by X^+ the free semigroup over X and by X^* — the free monoid over X with an empty word θ.

For $w = w_1 w_2 \dots w_n \in X^+$ we shall use the following notation:
$|w| = n$ — the number of letters occuring in w, $|\theta| = 0$;
$f \in X^*$ — a *prefix* (*suffix*, *factor*) of w if $w = fv$ ($w = vf$, $w = vfu$) for some $v, u \in X^*$;
$(w)_x^n = (wx)^{n-1}w$ — a *deformed power* of w ($x \in X^*, n \in \mathbb{N}$);
$f^{-1}w$ (wf^{-1}) — the word obtained from w by deleting a prefix (suffix) $f \in X^*$ if such one exists and $f^{-1}w = w$ ($wf^{-1} = w$) otherwise;
$w(i) = w_i$ — the ith letter of w;
$w(i,j) = w_i \dots w_j$ — the (i,j)-factor of w.

Later we shall need the following results:

Lemma 1.1. [7, P.390, Lemma 5.1] *Let $u, v, w \in X^+$, then $vu = uw$ if and only if*

$$\begin{array}{cc} v = ab, & w = ba. \\ u = (a)_b^n & (a \in X^+, b \in X^*, n \in \mathbb{N}). \end{array} \tag{1}$$

For $u, v, w \in X^+$ such that $vu = uw$ it follows from the previous lemma that

$$(|v| = |w| < |u|) \Leftrightarrow \begin{cases} u = (a)_b^n \\ v = ab, \\ w = ba, \end{cases} \quad (n > 1). \tag{2}$$

Given $u \in X^+$, we denote by $\deg(u)$ the maximal integer n such that $u = (a)_b^n$ for some $a \in X^+, b \in X^*$ and put $\deg(\theta) = 0$.

For u such that $\deg(u) > 1$ define the set $\Delta(u)$ as follows

$$\Delta(u) = \{(ba)^k \,:\, a \in X^+,\, b \in X^*, u = (a)_b^n \;\&\; 1 \le k < n \le \deg(u)\},$$

and $\Delta(u) = \varnothing$ otherwise.

I.e. $\Delta(u)$ contains all words of the form $(ba)^k$ ($a \in X^+$, $b \in X^*, 1 \le k < \deg(u)$) for which there exist $n \in \{k+1, \dots, \deg(u)\}$ such that $u = (a)_b^n$.

Now put $\widetilde{\theta} = \theta$ and

$$\widetilde{f} = (uf)u^{-1} \quad (f \in \Delta(u)), \qquad \widetilde{\Delta}(u) = \{\widetilde{f} \,:\, f \in \Delta(u)\}.$$

For all $v \in X^* \setminus (\Delta(u) \cup \widetilde{\Delta}(u))$ put $\widetilde{v} = v$. One may straightforwardly check, that for u with $\deg(u) > 1$ we have

$$\begin{aligned} \widetilde{\Delta}(u) \;=\; & \{\widetilde{(ba)^k} = (ab)^k \,:\, a \in X^+,\, b \in X^*, \\ & u = (a)_b^n \;\&\; 1 \le k < n \le \deg(u)\}. \end{aligned}$$

Example 1.2. Let $a, b, c, d \in X$ and consider next words:
1) $u = abcdb$. To obtain $\Delta(u)$ we should find all representations of u in form (1). But we have only $u = (u)_\theta^1$. Hence $\deg(u) = 1$ and $\Delta(u) = \varnothing$. In fact, we shall get these results if and only if there are no coinciding nonempty prefix and suffix of u except u itself.
2) $u = abababababa$. We have

$$u = (a)_b^6 = (aba)_b^3 = (aba)_{babab}^2 = (ababa)_b^2 = (a)_{bababababab}^2,$$

hence $\deg(u) = 6$. To find all elements of $\Delta(u)$ we should consider each case, which we obtained, using definition of $\Delta(u)$:

$$\begin{aligned} (a)_b^6 :&\quad ba, (ba)^2, (ba)^3, (ba)^4, (ba)^5, \\ (aba)_b^3 :&\quad b(aba) = (ba)^2, (b(aba))^2 = (ba)^4, \\ (aba)_{babab}^2 :&\quad (babab)(aba) = (ba)^4, \\ (ababa)_b^2 :&\quad b(ababa) = (ba)^3, \\ (a)_{bababababab}^2 :&\quad (bababababab)a = (ba)^5; \end{aligned}$$

$\Delta(u)$ is the set of all these elements, i.e.

$$\Delta(u) = \{ba, (ba)^2, (ba)^3, (ba)^4, (ba)^5\}.$$

Now let us find $\widetilde{\Delta}(u)$. We have

$$\widetilde{(ba)} = (uba)u^{-1} = (abu)u^{-1} = ab,$$

another cases are similar, so we have

$$\widetilde{\Delta}(u) = \{ab, (ab)^2, (ab)^3, (ab)^4, (ab)^5\}.$$

3) $u = abacaba$. Similarly to 2) we obtain $u = (aba)_c^2 = (a)_{bacab}^2$, and hence $\deg(u) = 2$, $\Delta(u) = \{c(aba), (bacab)a\}$.

The next proposition follows from Lemma 1.1:

Proposition 1.3. *Let $u, w \in X^+$. Then $w \in \Delta(u)$ if and only if there exists $v \in X^+$ such that $uw = vu$ and $|w| < |u|$. Moreover, $v = \widetilde{w}$.*

By the previous proposition, $\Delta(u)$ consists of all nonempty words d which are shorter than u and have the following property: $ud = \widetilde{d}u$. This simple and important fact will be frequently used in further proofs.

A word $w \in X^+$ is called *primitive* if

$$w = v^k \Rightarrow k = 1 \quad (v \in X^+, k \in \mathbb{N}).$$

Denote the set of all primitive words by $P(X)$. Further we shall need the following lemmas :

Lemma 1.4. [7, P.390, Corollary 5.3] *For all $x, y \in X^+$, equality $xy = yx$ implies $x = z^p$ and $y = z^q$ for some $z \in X^+, p, q \in \mathbb{N}$.*

Lemma 1.5. [7, P.391, Lemma 5.5] *Let $x, y \in X^+$. If there exist integers p, q and words $u, v, u', v', w \in X^*$ such that*

$$x^p = uwv,\ y^q = u'wv',$$

$$|w| \geq |x| + |y| - \mathrm{GCD}(|x|, |y|),$$

$$|v| \equiv |v'| \,(\mathrm{mod}\, \mathrm{GCD}(|x|, |y|)),$$

then x and y are powers of the same word $z \in X^+$.

Notice, that in previous lemma we may replace the condition $|v| \equiv |v'|$ by a dual one: $|u| \equiv |u'|$.

Lemma 1.6. [7, P.392, Proposition 5.6] *For any $w \in X^+$, there exist unique word $z \in P(X)$ and integer $p \in \mathbb{N}$ such that $w = z^p$.*

Lemma 1.7. *For any $w, v \in X^+$, if w is a power of v, then v is a power of $z \in P(X)$ such that $w = z^p$ $(p \in \mathbb{N})$.*

Proof. Let $w = v^n$. By Lemma 1.6 we have $w = z^p$ for unique $z \in P(X)$ and $p \in \mathbb{N}$. If $n = 1$ or $n = p$ there is nothing to prove. Assume that $1 < n < p$, then $|v| \leq \frac{|w|}{2}$ and $|z| \leq \frac{|w|}{3}$. Hence $|w| \geq |v| + |z|$ and, by Lemma 1.5, v and z are powers of the same word, i.e. v is a power of z, since z is primitive. □

Proposition 1.8. *Let $x = (a)_b^n$ with $n = \deg(x)$, $a \in X^+$, $b \in X^*$ and put*

$$\Delta_0(x) = \{dc : c \in X^+, d \in X^*, x = (c)_d^2\},$$

$$\Delta_1(x) = \begin{cases} \{(ba)^k : 1 \le k < n\}, & n > 2; \\ \varnothing, & n \le 2. \end{cases}$$

Then $ba, ab \in P(X)$ and $\Delta(x) = \Delta_0(x) \cup \Delta_1(x)$.

Proof. The case $\deg(x) < 2$ is trivial. Assume that $x = (a)_b^n$, where $n = \deg(x) \ge 2$; $a \in X^+$; $b \in X^*$ and the word ba is not primitive. Lemma 1.6 implies that $ba = v^l$ for some $v \in P(X)$ and $l > 1$. Then $a = gv^p$ $(p \ge 0)$ where g is a suffix of v. So $v = fg$ and hence

$$x \;=\; a(ba)^{n-1} = av^{l(n-1)} = g(fg)^{p+l(n-1)},$$

and n is not maximal, a contradiction. The proof of the inclusion $ab \in P(X)$ is dual.

Sets $\Delta_0(x)$ and $\Delta_1(x)$ are defined as subsets of $\Delta(x)$, so we have $\Delta_0(x) \cup \Delta_1(x) \subseteq \Delta(x)$. If $n = 2$, then $\Delta(x) = \Delta_0(x)$. For $n > 2$, suppose that $x = (c)_d^m$ and $(dc)^p \in \Delta(x) \setminus (\Delta_0(x) \cup \Delta_1(x))$ for some $c \in X^+; d \in X^*; 2 < m \le n; 1 \le p < m$ (we put $m > 2$ because $(dc)^p \in \Delta_0(x)$ if $m = 2$). We also have $|ba| \le |dc|$, because ba is the shortest word in $\Delta(x)$ since n is maximal.

Now let $\mathrm{GCD}(|ba|, |dc|) = k$. Using inequalities $m > 2$ and $|dc| \ge |ba|$ we obtain

$$|x| = (m-1)|dc| + |c| \ge 2|dc| + |c| \ge |dc| + |ba| - k.$$

Then, since $(ba)^n = bx\theta, (dc)^m = dx\theta$, we have $ba, dc \in z^*$ for some $z \in P(X)$ by Lemma 1.5. But $ba \in P(X)$, so $dc = (ba)^q$ for some $q \in \mathbb{N}$, thus $(dc)^p = (ba)^{pq}$. Hence $(dc)^p \in \Delta_1(x)$. Obtained contradiction implies $\Delta(x) \subseteq \Delta_0(x) \cup \Delta_1(x)$ and, finally, $\Delta(x) = \Delta_0(x) \cup \Delta_1(x)$. □

Example 1.9. Let $u = (a)_b^6$. We have $\deg(u) = 6$ and considering case $u = (a)_b^6$, by Example 1.2 (2), we obtain $\Delta(u) = \Delta_1(u)$.

Again, $\Delta_0(u)$ is the set of all words vw, where $w, v \in X^*$ such that $u = (w)_v^2$ $(w \ne \theta)$. Therefore, by Example 1.2 (2), $\Delta_0(u) = \{(ba)^3, (ba)^4, (ba)^5\}$.

Proposition 1.10. *If $x = u^k$ $(u \in P(X),\ k > 1)$, then $k = \deg(x)$.*

Proof. Let $x = u^k = (a)_b^n$ $(n = \deg(x))$ for some $a \in X^+; b \in X^*; u \in P(X); k > 1$. We have $k \le n$ by maximality of n. Assume that $k \ne n$,

i.e. $1 < k < n$. Then $|u| \geq |ba|$ because x contains at least $n - 1 \geq k$ occurrences of ba. Therefore,

$$(ba)^n = bx\theta, \quad u^k = x\theta, \quad |\theta| \equiv |\theta| (\mathrm{mod}\, \mathrm{GCD}(|u|, |ba|)).$$

$$|x| = k|u| \geq 2|u| \geq |u| + |ba| - \mathrm{GCD}(|u|, |ba|).$$

Hence Lemma 1.5 implies that u and ba are powers of the same word. By Proposition 1.8, we have $ba \in P(X)$, i.e. $u = ba$ since $u \in P(X)$. Therefore, $x = u^k = au^{n-1}$. But $n - 1 \geq k$ and $a \neq \theta$, a contradiction. Thus $n = k$. □

Proposition 1.11. *Let $x \in X^*$. If $\deg(x) \geq 3$, then there are unique $ba \in \Delta(x)$ such that $a \in X^+$, $b \in X^*$ and $x = (a)_b^{\deg(x)}$.*

Proof. Let $x = (a)_b^n$ for some $a \in X^+, b \in X^*$ and $n = \deg(x) \geq 3$. Now we shall prove the uniqueness of ba. Suppose that $x = (c)_d^n$, where $c \in X^+, d \in X^*$ and $ba \neq dc$. We may assume without loss that $|dc| > |ba|$. Then

$$(ba)^n = bx\theta, \quad (dc)^n = dx\theta, \quad |\theta| \equiv |\theta| (\mathrm{mod}\, \mathrm{GCD}(|ba|, |dc|)),$$

$$|x| > n|dc| \geq |dc| + |ba| - \mathrm{GCD}(|dc|, |ba|),$$

whence, by Lemma 1.5, words dc and ba are powers of the same primitive word and Proposition 1.8 implies that words dc and ba are primitive. Therefore, $ba = dc$ is unique. □

2. Certain representations of words

By A^* we denote the submonoid of X^* generated by $A \subseteq X^*$ and put $A^+ = A^* \setminus \{\theta\}$ (we shall write w^* instead of $\{w\}^*$). The set $A \subseteq X^*$ is called *a prefix code* if for any $a, b \in A$ the word a is a prefix of b iff $a = b$. Any prefix code A is the minimal generating set of the free submonoid A^* (see [8, P.5]).

Let $x \in X^+$ and s be the shortest word in $\Delta_1(x)$. We shall use the following notation:

$$D(x) = \begin{cases} (\Delta_0(x) \setminus \Delta_1(x)) \cup \{s\}, & \text{if } \Delta_1(x) \neq \varnothing, \\ \Delta_0(x), & \text{if } \Delta_1(x) = \varnothing. \end{cases}$$

and write $D^*(x)$ and $\Delta^*(x)$ instead of $(D(x))^*$ and $(\Delta(x))^*$ respectively. $D(x)$ has the following property:

Lemma 2.1. *If for $x \in X^+$ the set $D(x)$ is nonempty, then $D(x)$ is a prefix code and $D^*(x) = \Delta^*(x)$.*

Proof. Suppose that $x = a(ba)^n = cdc$ where $ba \in D(x)$, $dc \in \Delta_0(x) \setminus \Delta_1(x)$ and ba is a prefix of $dc \neq ba$. But ba is also a suffix of dc, since $a(ba)^n = cdc$. Thus $dc = pba = baq$ and Lemma 1.1 implies that

$$p = uv, \quad q = vu, \quad ba = (u)_v^m \quad (u \in X^+, v \in X^*, m \in \mathbb{N}).$$

Hence $dc = uvba = bavu$ and

$$x = a(ba)^n = cdc = cuvba \Rightarrow a = cuv,$$

then uv and vu are suffixes of x, i.e. $uv = vu$. Therefore, by Lemma 1.4, words u and v are powers of some $z \in P(X)$. We have $dc = z^l$ $(l > 1)$ and $c = fz^k$ $(k \geq 0)$ where f is a suffix of z. Let $z = gf$, then

$$x = cdc = f(gf)^{l+k} = (f)_g^{l+k+1},$$

whence $z \in \Delta_1(x)$ and so $dc \in \Delta_1(x)$. Obtained contradiction proves that $D(x)$ is a prefix code. By definition of sets $\Delta(x)$ and $D(x)$, any word from $\Delta(x)$ is a power of some word from $D(x)$ and $D(x) \subseteq \Delta(x)$, whence $D^*(x) = \Delta^*(x)$. □

For any word $w \in X^*$, we say that factors $w(a, b)$ and $w(c, d)$ overlap if $[a, b] \cap [c, d] \neq \varnothing$.

Lemma 2.2. *If $u \in X^*$ consists of overlapping factors x, i.e. for any $k \leq |u|$ there exist $i, j \in \mathbb{N}$ such that $i \leq k \leq j$ and factor $u(i, j) = x$ overlaps with some $u(i', j') = x$ $(i \neq i')$, then $u \in xD^+(x)$.*

Proof. Overlapping of two factors x has a form $xw = vx$ where $|w| < |x|$. Proposition 1.3 implies that $w \in \Delta(x)$ and $v = \widetilde{w}$.

If $u \in X^*$ consists of overlapping factors x, then $u \in X^*x$ and $u = w_1\widetilde{d_1}x = w_1xd_1$ for some $d_1 \in \Delta(x), w_1 \in X^*$. If $w_1 \neq \theta$, then we similarly obtain $u = w_2xd_2d_1$ and we may continue:

$$u = w_2xd_2d_1 = w_3xd_3d_2d_1 = \ldots = xd_n \ldots d_1, \quad (d_i \in \Delta(x))$$

thus $u \in xD^+(x)$. □

Further, for any $w \in X^*$, we may find all factors x of xwx including their overlaps and write xwx as a product

$$xwx = (xd_1)w_1(xd_2)w_2 \ldots (xd_n)w_n(xd_{n+1})$$

where $w_1, \ldots, w_n \in X^*$ and $d_1, \ldots, d_{n+1} \in D^*(x)$ by Lemma 2.2, then

$$wx = (d_1w_1) *_x (d_2w_2) *_x \ldots *_x (d_nw_n) *_x d_{n+1}. \tag{3}$$

Hence we may write w in the following form

$$w = ((d_1w_1) *_x (d_2w_2) *_x \ldots *_x (d_nw_n) *_x d_{n+1})x^{-1}. \tag{4}$$

For any $i \leq n$, we call w_i a *barrier* of w.

If there are words $w_i = w_{i+1} = \ldots = w_j = \theta$ $(i \leq j \leq n)$ in (4), then the following factor of w

$$(d_iw_i)x(d_{i+1}w_{i+1})x \ldots (d_jw_j)xd_{j+1} = (d_ix)(d_{i+1}x) \ldots (d_jx)d_{j+1}$$

is contained in the set $(D(x) \cup \{x\})^*$. So we may write any $w \in X^+$ as

$$w = ((e_1v_1) *_x (e_2v_2) *_x \ldots *_x (e_kv_k) *_x e_{k+1})x^{-1}, \tag{5}$$

where $e_1, \ldots, e_{n+1} \in (D(x) \cup \{x\})^*$ and $v_1, \ldots, v_k$ are nonempty barriers of w. Such representation of words will be used for studying of X_x^+.

Example 2.3. Let $a, b, c, d \in X$:
1) $x \in X$, $w = bcxxd$. Since $\Delta(x) = \varnothing$, we have $D^*(x) = \{\theta\}$. Then by (4) we obtain

$$w = ((bc) *_x \theta *_x d *_x \theta)x^{-1}.$$

where bc, θ and d are barriers of w.

For the form (5) we similarly have:

$$w = ((bc) *_x (xd) *_x \theta)x^{-1}.$$

2) $x = a^n$, $w = \theta$. We have $D^*(x) = a^*$ and $x\theta x = a^{2n}$. From (4) it follows that $\theta = a^nx^{-1}$ $(a^n \in D^*(x))$. This is the example of words which has no barriers.

3) Now consider more complicated example:

$$x = ababa, \quad w = bacbaababaddaababab.$$

We have $\Delta(x) = \{ba, baba\}$, $D(x) = \{ba\}$. Find all occurrences of x in xwx:

$$xwx = \underline{ababa}\underline{ba}cba\underline{ababa}\underline{ba}dda\underline{ababa}\underline{ba}\underline{ba}\underline{ba}.$$

Using the same notation as in (4) we obtain

$$w = ((d_1w_1) *_x (d_2w_2) *_x d_3)x^{-1} = ((ba)cba *_x dda *_x (bababa))x^{-1},$$

where $d_1 = ba,\ d_2 = \theta,\ d_3 = (ba)^3$ and $w_1 = cba,\ w_2 = dda$. Representation of w in form (5) is the same as in (4), because w does not have empty barriers.

Proposition 2.4. *Denote the set of all words b, such that b is a barrier of some $w \in X_x^*$, by* $\mathrm{Bar}(x)$. *For any $x \in X^+$,*

$$\mathrm{Bar}(x) = \\ = \begin{cases} X^* \setminus \left((X_x^*)^2 \cup (\Delta(x)X^*)x^{-1} \cup X^*\widetilde{\Delta}(x)\right), & \text{if } \deg(x) > 1; \\ X^* \setminus (X_x^*)^2, & \text{otherwise.} \end{cases}$$

Proof. Let $\deg(x) \leq 1$, hence $\Delta(x) = \varnothing$ and $\mathrm{Bar}(x) = \{\theta\}$, so the set $\mathrm{Bar}(x)$ consists of all words not containing x as a factor, i.e. $\mathrm{Bar}(x) = X^* \setminus (X_x^*)^2$.

Now assume that $\deg(x) > 1$. If $w \in X^* \setminus ((X_x^*)^2 \cup (\Delta(x)X^*)x^{-1} \cup X^*\widetilde{\Delta}(x))$, then xwx has only two factors x, since wx has no prefixes from $\Delta(x)$, xw has no suffixes from $\widetilde{\Delta}(x)$ and x is not a factor of w. Therefore, the form (4) of w is $(w *_x \theta)x^{-1}$ and w is a barrier of itself:

$$X^* \setminus \left((X_x^*)^2 \cup (\Delta(x)X^*)x^{-1} \cup X^*\widetilde{\Delta}(x)\right) \subseteq \mathrm{Bar}(x).$$

Consider some barrier w_i of an arbitrary word w in notation of (4). The factor xw_ix of xwx does not contain overlapping occurrences of x. Obviously, $w_i \notin (X_x^*)^2$. If $w_i \in (\Delta(x)X^*)x^{-1}$, then $w_ix = dv$ for some $v \in X^*, d \in \Delta(x)$ and

$$xw_ix = xdv = \widetilde{d}xv$$

hence xw_ix has overlapping factors $xw_ix(1, |x|) = x$ and $xw_ix(|d| + 1, |dx|) = x$, a contradiction. Similarly, for $w_i \in X^*\widetilde{\Delta}(x)$ we have $w = v\widetilde{d}$ and $xwx = xv\widetilde{d}x = xvxd$. Thus,

$$\mathrm{Bar}(x) = X^* \setminus \left((X_x^*)^2 \cup (\Delta(x)X^*)x^{-1} \cup X^*\widetilde{\Delta}(x)\right)$$

as required. □

3. Variants of free monoids

Let X and Y be nonempty sets and $x \in X^*, y \in Y^*$. Since X_θ^* contains an identity, it is obvious that X_θ^* and Y_y^* are isomorphic if and only if y is empty and $X^* \cong Y^*$. If $X = \{a\}$, then variants of a^* are commutative and pairwise nonisomorphic [3, Theorem 1.1]. Hence

$$\{a\}_{a^n}^* \cong Y_y^* \text{ iff } Y = \{b\} \text{ and } y = b^n.$$

Thats why we put $|X| > 1$, $|Y| > 1$ and fix $x \in X^*, y \in Y^*$ such that $x \neq \theta, y \neq \theta'$ where $\theta' \in Y^*$ is an empty word of Y^*.

Lemma 3.1. *If $w \in \mathrm{Bar}(x)$ ($w \in \mathrm{Bar}(x) \setminus \{\theta\}$) and $\phi : X_x^* \to Y_y^*$ ($\phi : X_x^+ \to Y_y^+$) is an isomorphism, then $\phi(w) \in \mathrm{Bar}(y)$.*

Proof. Let $w \in \mathrm{Bar}(y) \setminus \{\theta\}$ and $\phi : X_x^+ \to Y_y^+$ (or $\phi : X_x^* \to Y_y^*$) be an isomorphism.

Suppose that $\phi(w) \notin \mathrm{Bar}(y)$, then by Proposition 2.4 we have $\phi(w) \in (Y_y^*)^2 \cup (\Delta(y)Y^*)y^{-1} \cup Y^*\widetilde{\Delta}(y)$ which implies that at least one of the next equalities hold

$$\begin{aligned}
\phi(w) &= u *_y v \in (Y_y^*)^2,\\
\phi(w) &= v\tilde{d} \in Y^*\widetilde{\Delta}(y),\\
\phi(w) &= dv \in \Delta(y)Y^*,\\
\phi(w) &= (d_1 \dots d_k)y^{-1} \in (\Delta(y)Y^*)y^{-1} \setminus \Delta(y)Y^*
\end{aligned}$$

for some $u, v \in Y^*$; $d, d_1, \dots, d_k \in \Delta(y)$ where $d_1 \dots d_k = ay$ ($a \in Y^+$). By definition of a barrier and Proposition 2.4 it follows that w is a nonempty barrier of $w *_x w *_x w$, hence if $w *_x w *_x w = f *_x g$ for some $f, g \in X^+$, then $g = w$ or $f = w$, because we have an overlapping occurrences of x in $w *_x w *_x w$ oterwise. But, as showed above, for the word $(\phi(w))_y^3$ at least one of the next additional equalities hold

$$\begin{aligned}
(\phi(w))_y^3 &= (dv)_y^3 = (dv\tilde{d}) *_y (vydv),\\
(\phi(w))_y^3 &= (v\tilde{d})_y^3 = (v\tilde{d}yv) *_y (dv\tilde{d}),\\
(\phi(w))_y^3 &= (u *_y v)_y^3 = (u *_y v *_y u) *_y (v *_y u *_y v),\\
(\phi(w))_y^3 &= (a)_y^3 = a *_y d_1 \dots d_k a = (a\tilde{d_1}) *_y (d_2 \dots d_k a).
\end{aligned}$$

Therefore, we have $(w)_x^3 = f *_x g$ for some $f \neq w$ and $g \neq w$, since $(\phi(w))_y^3 = \phi(w *_x w *_x w)$. Obtained contradiction implies that $\phi(w) \in \mathrm{Bar}(y)$ as required. For empty barriers the proof is analogous. □

Assume that $\phi : X_x^+ \to Y_y^+$ is an isomorphism. By the previous lemma we can fix (for further using) words $p \in \mathrm{Bar}(x) \setminus \{\theta\}$ and $q \in \mathrm{Bar}(y) \setminus \{\theta'\}$ such that $\phi(p) = q$.

For the isomorphism $\phi^* : X_x^* \to Y_y^*$ we similarly fix words $p' \in \mathrm{Bar}(x)$ and $q' \in \mathrm{Bar}(x)$ such that $\phi^*(p') = q'$.

Now we are ready to formulate isomorphism criteria:

Lemma 3.2. *If $X_x^* \cong Y_y^*$ or $X_x^+ \cong Y_y^+$, then $|\Delta(x)| = |\Delta(y)|$.*

Proof. Assume that $|\Delta(x)| > |\Delta(y)|$. Since $\phi : X_x^+ \to Y_y^+$ is an isomorphism, there exists $d \in \Delta(x) \cup \{x\}$ such that $\phi(dp) \notin (\Delta(y) \cup \{y\})q$. By Proposition 1.3 we have:

$$\phi(p *_x dp) = \phi(p\tilde{d} *_x p) = \phi(p\tilde{d}) *_y \phi(p) = \phi(p\tilde{d}) *_y q,$$

and on the other hand

$$\phi(p *_x dp) = \phi(p) *_y \phi(dp) = q *_y \phi(dp).$$

Equality $\phi(p\tilde{d}) *_y q = q *_y \phi(dp)$ implies that $\phi(dp) = eq$ and $\phi(p\tilde{d}) = qe'$ for some $e, e' \in Y^+$. Proposition 2.4 implies that there is no factors x in dp and $p\tilde{d}$ and hence there is no factors y in eq and qe'. By reduction of q in $qe' *_y q = q *_y eq$ we obtain $e'y = ye$.

Thus, according to (2) and Lemma 1.1, we have $e = wy$ for some $w \in X^+$ since $\phi(dp) \notin (\Delta(y) \cup \{y\})q$. So $\phi(dp) = w *_y q$ and hence $\phi(dp) = \phi(w'xp)$ where $w' = \phi^{-1}(w) \in X^+$. Therefore, $d = w'x$, but $|d| \leq |x|$. Obtained contradiction proves that $\phi(dp) \in \Delta(y)q$ and hence $|\Delta(x)| \leq |\Delta(y)|$.

The proof of inequality $|\Delta(x)| \geq |\Delta(y)|$ is dual, so $|\Delta(x)| = |\Delta(y)|$ as required.

For the case $X_x^* \cong Y_y^*$, the proof is analogous. We assume that $|\Delta(x)| > |\Delta(y)|$, but consider $d \in \Delta(x)$ such that $\phi^*(dp') \notin (\Delta(y))q'$. And similarly obtain $d = w'x$ for some $w' \in X^*$ which contradicts with $|d| < |x|$. □

The proof of the previous lemma implies that

$$\phi((\Delta(x) \cup \{x\})p) = (\Delta(y) \cup \{y\})q, \tag{6}$$

$$\phi^*(\Delta(x)p') = \Delta(y)q'. \tag{6'}$$

Let $\phi(dp) = eq$ for some $d \in \Delta(x) \cup \{x\}, e \in \Delta(y) \cup \{y\}$, then

$$\begin{array}{rcl} \phi(p *_x dp) & = & q *_y eq = q\tilde{e} *_y q, \\ \phi(p *_x dp) & = & \phi(p\tilde{d} *_x p) = \phi(p\tilde{d}) *_y q, \end{array}$$

hence

$$\phi(p\tilde{d}) = q\tilde{e}. \tag{7}$$

And for $d' \in \Delta(x), e' \in \Delta(y)$, such that $\phi^*(d'p') = e'q'$, we similarly obtain

$$\phi^*(p'\tilde{d'}) = q'\tilde{e'}. \tag{7'}$$

Lemma 3.3. *If* $X_x^* \cong Y_y^*$, *then* $\deg(x) = \deg(y)$.

Proof. Let $\phi^* : X_x^* \to Y_y^*$ be an isomorphism. Assume that $\deg(x) = n < \deg(y) = m$ and $y = (a)_b^m$ $(a \in Y^+, b \in Y^*)$. By (6′) there exists $d \in \Delta(x)$ such that $\phi^*(dp) = baq'$. Suppose that $\phi^*(d^kp') = (ba)^kq'$ for some $k \in \mathbb{N}$, then

$$\begin{array}{rcl} \phi^*(p'\tilde{d} *_x d^kp') & \overset{(7')}{=} & q'ab *_y (ba)^kq' = q' *_y (ba)^{k+1}q', \\ \phi^*(p'\tilde{d} *_x d^kp') & = & \phi^*(p' *_x d^{k+1}p') = q' *_y \phi^*(d^{k+1}q') \end{array}$$

which imply, by induction, that

$$\phi^*(d^{k+1}p') = (ba)^{k+1}q'. \tag{8}$$

By (8) we have $\phi^*(d^n p') = (ba)^n q'$. Since $d = (fe)^s \in \Delta(x)$ for some $e \in X^+$, $f \in X^*$ such that $x = (e)_f^t$ $(s < t \leq n)$, we obtain

$$\begin{aligned}\phi^*((fe)^{sn}p') &= \phi^*((fe)^{sn-t}f(e)_f^t p') \\ &= \phi^*((fe)^{sn-t}f) *_y q' \in (Y_y^*)^2. \end{aligned} \tag{9}$$

But $|(ba)^n| \leq |y|$ and hence $(ba)^n q' \in Y_y^+ \setminus (Y_y^*)^2$. Obtained contradiction implies $n \geq m$.

The proof of the inequality $n \leq m$ is dual and hence $\deg(x) = \deg(y)$. □

The following lemma gives an important consequence from the previous proof.

Lemma 3.4. *Let $X_x^* \cong Y_y^*$ with $x = (u)_v^n$ $(u \in X^+, v \in X^*)$ and $y = (a)_b^n$ $(a \in Y^+, b \in Y^*)$ where $n = \deg(x) = \deg(y) > 2$, then*

$$\phi(vup) = baq.$$

Proof. According to (6′), we have $\phi(vup) = dq$ for some $d \in \Delta(y)$ and hence

$$\phi((vu)^{n-1}p) \overset{(8)}{=} d^{n-1}q.$$

By isomorphism $X^* \cong Y^*$, inclusion $(vu)^{n-1}p \in X^* \setminus (X^*)^2$ implies $d^{n-1}q \in Y^* \setminus (Y^*)^2$. Therefore, $|d^{n-1}| < |x|$ and so $d = b'a'$, where $a' \in X^+, b' \in X^*$ are such that $y = (a')_{b'}^k$ with $k \geq n$. Maximality of n implies $k = n$ and, by uniqueness of $b'a'$ (Proposition 1.11), we have $b'a' = ba$. □

Theorem 3.5. *For all $x \in X^+$ and $y \in Y^+$, variants X_x^* and Y_y^* are isomorphic if and only if the following equalities hold:*

(i) $|\Delta(x)| = |\Delta(y)|$,

(ii) $\deg(x) = \deg(y)$.

Proof. Lemmas 3.2 and 3.3 imply necessity. For sufficiency suppose that (i) and (ii) are held.

From Proposition 1.8 it follows that

$$\Delta(x) = \Delta_0(x) \cup \Delta_1(x) = (\Delta_0(x) \setminus \Delta_1(x)) \cup \Delta_1(x).$$

Hence, by (i) and $|\Delta_1(x)| = \deg(x) - 1 = |\Delta_1(y)|$, we have

$$|\Delta_0(x) \setminus \Delta_1(x)| = |\Delta_0(y) \setminus \Delta_1(y)| \Rightarrow |D(x)| = |D(y)|.$$

If $\Delta_1(x) \neq \varnothing$ and $\Delta_1(y) \neq \varnothing$, then let s and t be the shortest words from $\Delta_1(x)$ and $\Delta_1(y)$ respectively, put $s = \theta$, $t = \theta'$ otherwise. Let $\alpha : D(x) \cup \{\theta\} \to D(y) \cup \{\theta'\}$ be a bijection such that

$$\alpha(s) = t, \qquad \alpha(\theta) = \theta', \tag{10}$$

and α may be extended to the isomorphism $\alpha : D^*(x) \to D^*(y)$ by

$$\alpha(a_1 \dots a_k) = \alpha(a_1) \dots \alpha(a_k) \quad (a_1, \dots, a_k \in D(x)). \tag{11}$$

Proposition 2.4 implies that sets $\mathrm{Bar}(x)$ and $\mathrm{Bar}(y)$ are both countably infinite and so there exists some bijection $\beta : \mathrm{Bar}(x) \to \mathrm{Bar}(y)$.

For all $w = ((d_1 w_1) *_x \dots *_x (d_n w_n) *_x d_{n+1}) x^{-1}$, where $d_i \in D^*(x)$ and $w_j \in \mathrm{Bar}(x)$, define $\psi : X_x^* \to Y_y^*$ by putting

$$\psi(w) = (\alpha(d_1)\beta(w_1) *_y \dots *_y \alpha(d_n)\beta(w_n) *_y \alpha(d_{n+1})) y^{-1}. \tag{12}$$

Words $\beta(w_1), \dots, \beta(w_n)$ are barriers of $\psi(w)$ and right part of (17) coincides with the form (4) of $\psi(w)$.

Images of α and β coincide with $D^*(y)$ and $\mathrm{Bar}(y)$ respectively. Therefore, $\operatorname{Im}\psi$ contains all elements from Y_y^* and hence ψ is surjective.

Let $v = ((e_1 v_1) *_x \dots *_x (e_m v_m) *_x e_{m+1}) x^{-1}$, where $e_i \in D^*(x)$ and $v_j \in \mathrm{Bar}(x)$. If $v \neq w$, then $d_i \neq e_i$ or $w_j \neq v_j$ for some $i, j \in \mathbb{N}$ and hence $\psi(v) \neq \psi(w)$. Therefore, ψ is injective.

Further,

$$\begin{aligned}
\psi(w *_x v) &= \psi((d_1 w_1 *_x \dots *_x d_n w_n *_x d_{n+1} e_1 v_1 \\
&*_x \ e_2 v_2 *_x \dots *_x e_m v_m *_x e_{m+1}) x^{-1}) \\
&= (\alpha(d_1)\beta(w_1) *_y \dots *_y \alpha(d_n)\beta(w_n) *_y \alpha(d_{n+1} e_1)\beta(v_1) \\
&*_y \ \dots *_y \alpha(e_m)\beta(v_m) *_y \alpha(e_{m+1})) y^{-1} \\
&= (\alpha(d_1)\beta(w_1) *_y \dots *_y \alpha(d_n)\beta(w_n) *_y \alpha(d_{n+1})) y^{-1} \\
&*_y \ (\alpha(e_1)\beta(v_1) *_y \dots *_y \alpha(e_m)\beta(v_m) *_y \alpha(e_{m+1})) y^{-1} \\
&= \psi(w) *_y \psi(v)
\end{aligned}$$

Hence ψ is an isomorphism and so $X_x^* \cong Y_y^*$. □

The following corollary describes all isomorphisms between X_x^* and Y_y^*:

Corollary 3.6. *If $x \in X^+$ and $y \in Y^+$ are such that $X_x^* \cong Y_y^*$, then each isomorphism $\phi : X_x^* \to Y_y^*$ is as in* (17).

Proof. Assume that $\phi : X_x^* \to Y_y^*$ is an isomorphism. Let $d_1, \ldots, d_k \in D(x) \cup \{\theta\}$, $v \in \mathrm{Bar}(x)$. By (6'), (7') and (8) we have

$$\phi(d_i v) = e_i \phi(v), \quad \phi(v\tilde{d}_i) = \phi(v)\tilde{e}_i,$$

for some $e_i \in \Delta(y)$, $1 \leq i \leq k$. Suppose that

$$\phi(d_{k-1} \ldots d_1 v) = e_{k-1} \ldots e_1 \phi(v). \tag{13}$$

Thus

$$\begin{aligned} \phi(v) *_y \phi(d_k \ldots d_1 v) &= \phi(v *_x d_k \ldots d_1 v) \\ &= \phi(v\tilde{d}_k *_x d_{k-1} \ldots d_1 v) \\ &= \phi(v\tilde{d}_k) *_y \phi(d_{k-1} \ldots d_1 v) \\ &\overset{(7'),(13)}{=} \phi(v)\tilde{e}_k *_y e_{k-1} \ldots e_1 \phi(v) \\ &= \phi(v) *_y e_k \ldots e_1 \phi(v) \end{aligned}$$

and hence, by induction,

$$\phi(d_k \ldots d_1 v) = e_k \ldots e_1 \phi(v). \tag{14}$$

By (8) and (14) we obtain

$$\phi(d_k \ldots d_1 v) = \alpha(d_k) \ldots \alpha(d_1) \phi(v), \tag{15}$$

where $\phi|_{\mathrm{Bar}(x)}$ is a bijection by Lemma 3.1 and $\alpha : D(x) \cup \{\theta\} \to D(y) \cup \{\theta'\}$ is a bijection since ϕ is an isomorphism. We have $\alpha(\theta) = \theta'$ by (6'), also, according to Lemma 3.4, α sends the shortest word from $\Delta(x)$ to the shortest word from $\Delta(y)$ and so α is as in (10). We extend α to $\alpha : D^*(x) \to D^*(y)$ by (11).

If $d_k \ldots d_1 \in X^* x$, then

$$\begin{aligned} \phi((d_k \ldots d_1)x^{-1} *_x v) &= \phi((d_k \ldots d_1)v) \\ &\overset{(15)}{=} (\alpha(d_k) \ldots \alpha(d_1))\phi(v) \\ &= (\alpha(d_k) \ldots \alpha(d_1))y^{-1} *_y \phi(v), \end{aligned}$$

thus

$$\phi((d_k \ldots d_1)x^{-1}) = (\alpha(d_k) \ldots \alpha(d_1))y^{-1}. \tag{16}$$

For any $w = ((d_1w_1) *_x \ldots *_x (d_nw_n) *_x d_{n+1})x^{-1}$, where $d_i \in D^*(x)$, $w_j \in \mathrm{Bar}(x)$ and $n \geq 1$, we have

$$\begin{aligned}\phi(w *_x v) &= \phi(d_1w_1 *_x \ldots *_x d_nw_n *_x d_{n+1}v)\\ &\overset{(15)}{=} \alpha(d_1)\phi(w_1) *_y \ldots *_y \alpha(d_n)\phi(w_n) *_y \alpha(d_{n+1})\phi(v)\\ &= (\alpha(d_1)\phi(w_1) *_y \ldots *_y \alpha(d_n)\phi(w_n) *_y \alpha(d_{n+1}))y^{-1} *_y \phi(v).\end{aligned}$$

Therefore,

$$\phi(w) = (\alpha(d_1)\phi(w_1) *_y \ldots *_y \alpha(d_n)\phi(w_n) *_y \alpha(d_{n+1}))y^{-1}$$

and this result and (16) imply that ϕ is as in (17). □

Denote by $\mathcal{S}(X)$ and $\mathrm{Aut}(X_x^*)$ the symmetric group over the set X and the automorphism group of X_x^* respectively. As a consequence, we obtain the following theorem.

Theorem 3.7. *For all $x \in X^+$,*

$$\mathrm{Aut}(X_x^*) \cong \begin{cases} \mathcal{S}(\mathrm{Bar}(x)), & \Delta_0(x) \setminus \Delta_1(x) = \varnothing; \\ \mathcal{S}(\Delta_0(x) \setminus \Delta_1(x)) \times \mathcal{S}(\mathrm{Bar}(x)), & \Delta_0(x) \setminus \Delta_1(x) \neq \varnothing. \end{cases}$$

Proof. Assume that $\Delta_0(x) \setminus \Delta_1(x) \neq \varnothing$. For all $\alpha \in \mathcal{S}(\Delta_0(x) \setminus \Delta_1(x))$, define a transformation $\alpha^* : D^*(x) \to D^*(x)$ by the following

$$\forall u \in D(x) : \alpha^*(u) = \begin{cases} \alpha(u), & \text{if } u \in \Delta_0(x) \setminus \Delta_1(x); \\ u, & \text{otherwise;} \end{cases}$$

and

$$\forall a_1, \ldots, a_k \in D(x) : \alpha^*(a_1 \ldots a_k) = \alpha^*(a_1) \ldots \alpha^*(a_k),$$

For all $\beta \in \mathcal{S}(\mathrm{Bar}(x))$ and $w = ((d_1w_1) *_x \ldots *_x (d_nw_n) *_x d_{n+1})x^{-1}$, where $d_i \in D^*(x)$ and $w_j \in \mathrm{Bar}(x)$, we define the following maps:

$$\phi_{\alpha,\beta} = (\alpha^*(d_1)\beta(w_1) *_y \ldots *_y \alpha^*(d_n)\beta(w_n) *_y \alpha^*(d_{n+1}))y^{-1}. \qquad (17)$$

By Corollary 3.6, maps $\phi_{\alpha,\beta}$ are automorphisms. Corollary 3.6 also implies that

$$\mathrm{Aut}(X_x^*) = \{\phi_{\alpha,\beta} \,:\, \alpha \in \mathcal{S}(\Delta_0(x) \setminus \Delta_1(x)), \beta \in \mathcal{S}(\mathrm{Bar}(x))\}$$

and if $\alpha \neq \gamma$ or $\beta \neq \delta$, then $\phi_{\alpha,\beta} \neq \phi_{\gamma,\delta}$.

Thus the following map is a bijection:

$$\Phi : \mathcal{S}(\Delta_0(x) \setminus \Delta_1(x)) \times \mathcal{S}(\mathrm{Bar}(x)) \to \mathrm{Aut}(X_x^*) : (\alpha, \beta) \mapsto \phi_{\alpha,\beta}.$$

For all $\alpha, \gamma \in \mathcal{S}(\Delta_0(x) \setminus \Delta_1(x)), \beta, \delta \in \mathcal{S}(\mathrm{Bar}(x))$, we have

$$\begin{aligned}(\Phi(\alpha,\beta)\Phi(\gamma,\delta))(w) &= \phi_{\alpha,\beta}\phi_{\gamma,\delta}(w)\\ &= \phi_{\alpha,\beta}(\gamma^*(d_1)\delta(w_1)^*_y \ldots *_y \gamma^*(d_n)\delta(w_n) *_y \gamma^*(d_{n+1}))y^{-1})\\ &= \alpha^*\gamma^*(d_1)\delta(w_1) *_y \ldots *_y \alpha^*\gamma^*(d_n)\delta(w_n) *_y \alpha^*\gamma^*(d_{n+1}))y^{-1}\\ &= \Phi(\alpha\gamma, \beta\delta)(w) = \Phi((\alpha,\beta)(\gamma,\delta))(w).\end{aligned}$$

Therefore,

$$\Phi((\alpha,\beta)(\gamma,\delta)) = \Phi(\alpha,\beta)\Phi(\gamma,\delta)$$

and hence Φ is an isomorphism as required. The case $\Delta_0(x) \setminus \Delta_1(x) = \varnothing$ is similar. □

4. Variants of free semigroups

It this section we establish isomorphism criteria for variants of free semigroups.

Lemma 4.1. *If $X_x^+ \cong Y_y^+$, then the following implications hold:*

(i) $x \in P(X), y \in P(Y) \Rightarrow \deg(x) = \deg(y)$,

(ii) $x \notin P(X), y \notin P(Y) \Rightarrow \deg(x) = \deg(y)$,

(iii) $x \in P(X), y \notin P(Y) \Rightarrow \deg(x) = \deg(y) + 1$,

(iv) $x \notin P(X), y \in P(Y) \Rightarrow \deg(x) + 1 = \deg(y)$.

Proof. Let $\phi : X_x^+ \to Y_y^+$ be an isomorphism and $\deg(x) = n$, $\deg(y) = m$ and $y = (a)_b^m$ ($a \in Y^+, b \in Y^*$). By (6) there exists $d \in \Delta(x) \cup \{x\}$ such that $\phi(dp) = baq$. Suppose that $\phi(d^k p) = (ba)^k q$ for some $k \in \mathbb{N}$, then

$$\begin{aligned}\phi(p\tilde{d} *_x d^k p) &\overset{(7)}{=} qab *_y (ba)^k q = q *_y (ba)^{k+1} q,\\ \phi(p\tilde{d} *_x d^k p) &= \phi(p *_x d^{k+1} p) = q *_y \phi(d^{k+1} q)\end{aligned}$$

which imply that

$$\phi(d^{k+1}p) = (ba)^{k+1}q. \tag{18}$$

And, by induction, (18) holds for any $k \in \mathbb{N}^0$.

(i) By (18) we have $\phi(d^n p) = (ba)^n q$. Suppose that $x \in P(X), y = (a)_b^m \in P(Y)$ and $n < m$. We also have $n > 1$, since $|\Delta(x)| = |\Delta(y)| > 0$ by Lemma 3.2.

Consider the next cases:

1) If $d = x$, then $\phi(x^n p) = \phi(x^{n-1}) *_y p \in (Y_y^+)^2$. But $|(ba)^n| \leq |y|$ and

hence $\phi(x^n p) = (ba)^n q \notin (Y_y^+)^2$. Obtained contradiction implies $n \geq m$.
2) Now assume that $d = (fe)^s \in \Delta(x)$ for some $e, f \in X^+$, $s \in \mathbb{N}$ ($f \neq \theta$ since $x \in P(X)$) such that $x = (e)_f^t$ ($s < t \leq n$). Again, we have $\phi(x^n p) = (ba)^n q \notin (Y_y^+)^2$ and

$$\begin{aligned} \phi(d^n p) &= \phi((fe)^{sn} p) = \phi((fe)^{sn-t} f (e)_f^t p) \\ &= \phi((fe)^{sn-t} f *_x p) = \phi((fe)^{sn-t} f) *_y q \in (Y_y^+)^2, \end{aligned}$$

that is a contradiction as in case 1), so $n \geq m$.

Proof of the inequality $n \leq m$ is dual and hence $\deg(x) = \deg(y)$.

(ii) Suppose that $x \notin P(X), y = a^m \notin P(Y)$ and $n < m$. We have $|d^m| > |x| \Rightarrow d^m p \in (X_x^+)^2$ and hence $\phi(d^m p) \in (Y^+)^2$. On the other hand $\phi(d^m p) = a^m q = yq \notin (Y_y^+)^2$, a contradiction. Thus, $n \geq m$ and we dually obtain $n \leq m$ and $n = m$.

(iii) Suppose that $x \in P(X), y = a^m \notin P(Y)$ and $n < m + 1$. Again, we have

$$|d^n| > |x| \Rightarrow d^n p \in (X_x^+)^2 \Rightarrow \phi(d^n p) \in (Y_y^+)^2.$$

But $\phi(d^n p) = a^n q \notin (Y_x^+)^2$. Therefore, $n \geq m + 1$.

Now suppose that $n > m + 1$, then

$$\phi(d^{m+1} p) = a^{m+1} q = a *_y q \in (Y_y^+)^2.$$

On the other hand

$$|d^{m+1}| < |x| \Rightarrow d^{m+1} p \notin (X_x^+)^2 \Rightarrow \phi(d^{m+1} p) \notin (Y_y^+)^2.$$

Thus, $n \leq m + 1$ and, finally, $n = m + 1$.

(iv) This case is dual to (iii). □

Theorem 4.2. *For all $x \in X^+$ and $y \in Y^+$, variants X_x^+ and Y_y^+ are isomorphic if and only if $|\Delta(x)| = |\Delta(y)|$ and one of the following conditions is satisfied:*

(i) $x \in P(X), y \in P(Y)$ *and* $\deg(x) = \deg(y)$,

(ii) $x \notin P(X), y \notin P(Y)$ *and* $\deg(x) = \deg(y)$,

(iii) $x \in P(X), y \notin P(Y)$ *and* $\deg(x) = \deg(y) + 1$,

(iv) $x \notin P(X), y \in P(Y)$ *and* $\deg(x) + 1 = \deg(y)$.

Proof. Necessity follows from Lemmas 3.2 and 4.1. Let us prove the sufficiency.

(i–ii) Let $|\Delta(x)| = |\Delta(y)|$, $\deg(x) = \deg(y)$ and $x \in P(X)$, $y \in P(Y)$, then $X_x^* \cong Y_y^*$ by Theorem 3.5. Assume that $x\theta x$ has overlapping occurrences of x, then

$$\begin{cases} x\theta x = abxw = xbaw, \\ x = (a)_b^n, \end{cases} \qquad (a \in X^+;\, b, w \in X^*;\, n > 1)$$

and hence $ab = ba$ which implies, by Lemma 1.4, that there exist $z \in P(X)$ such that $a, b \in z^*$. Thus

$$x = (a)_b^n = z^k.$$

By $a \in X^+$ and $n > 1$ we have $k > 1$. Hence $x \notin P(X)$, a contradiction which proves that $x\theta x$ has no overlapping factors x and the form (4) of θ is $(\theta\theta *_x \theta)x^{-1}$. From the previous proof it follows that ba is not a prefix of $x \in P(X)$ and hence, by Lemma 2.1, $D(x) \cup \{x\}$ is a prefix code.

Further, for the map $\psi : X_x^* \to Y_y^*$ from (17), where β is such that $\theta \in \mathrm{Bar}(x)$ implies that $\beta(\theta) = \theta'$ (such map can be chosen since β is an arbitrary bijection), we have

$$\psi(\theta) = \psi((\theta\theta *_x \theta)x^{-1}) = (\alpha(\theta)\beta(\theta) *_y \alpha(\theta))y^{-1} = (\theta' *_y \theta')y^{-1} = \theta'.$$

Now let $|\Delta(x)| = |\Delta(y)|$, $\deg(x) = \deg(y) = n$ and $x \notin P(X)$, $y \notin P(Y)$. By Proposition 1.10, $x = v^n$ and $y = w^n$ for some $v \in X^+$, $w \in Y^+$. Then $\alpha(v) = w$ and θ can be represented in the form (4) by Example 2.3 as follows: $\theta = v^n x^{-1}$. Hence

$$\psi(\theta) = \psi(v^n x^{-1}) = \alpha(v)^n y^{-1} = w^n y^{-1} = \theta'.$$

Therefore, conditions $|\Delta(x)| = |\Delta(y)|$, $\deg(x) = \deg(y)$ and $(x \in P(X) \Leftrightarrow y \in P(X))$ imply that $\psi(\theta) = \theta'$. Thus $\psi(X^+) = Y^+$ and so $\psi|_{X^+}$ is an isomorphism from X_x^+ onto Y_y^+.

(iii) Suppose that condition (iii) holds and $y = t^m$ $(m > 1,\ t \in P(Y))$ and s is the shortest word from $\Delta_1(x)$.

We build isomorphism in the same way as in (17). The set $\Delta_1(x)$ is nonempty, since $y \notin P(Y)$ and $\deg(x) > \deg(y)$.

According to Proposition 1.8, we obtain

$$\begin{aligned} |D(x)| &= 1 + |\Delta(x)| - |\Delta_1(x)| \\ &= 1 + |\Delta(y)| - (\deg(x) - 1) \\ &= 1 + |\Delta(y)| - \deg(y) \\ &= |\Delta(y)| - |\Delta_1(y)| = |\Delta_0(y) \setminus \Delta_1(y)|, \end{aligned}$$

hence $|D(x) \cup \{x\}| = |D(y)|$.

Now let $\alpha : D(x) \cup \{x, \theta\} \to D(y) \cup \{\theta\}$ be a bijection such that $\alpha(s) = t,\ \alpha(\theta) = \theta'$ and put

$$\forall a_1, \dots, a_k \in D(x) \cup \{x, \theta\} : \alpha(a_1 \dots a_k) = \alpha(a_1) \dots \alpha(a_k).$$

We write an arbitrary word $w \in X^+$ as in (5):

$$w = ((e_1 v_1) *_x (e_2 v_2) *_x \dots *_x (e_k v_k) *_x e_{k+1}) x^{-1},$$

where $e_i \in (D(x) \cup \{x\})^*$ and $v_j \in \mathrm{Bar}(x) \setminus \{\theta\}$.

For a given bijection $\beta : \mathrm{Bar}(x) \to \mathrm{Bar}(y)$, we define $\xi : X_x^+ \to Y_y^+$ by the following formula

$$\xi(w) = (\alpha(e_1)\beta(v_1) *_y \dots *_y \alpha(e_n)\beta(v_n) *_y \alpha(e_{n+1})) y^{-1}. \qquad (19)$$

The remaining part of the proof is similar to the proof of Theorem 3.5.

(iv) We may build an isomorphism $Y_y^+ \to X_x^+$ in the same way as in (19). □

Example 4.3. 1) Consider the following words: $x = abcabcabc$ and $y = abcabcab$, where $a, b, c \in X$. We have $x = (abc)^3$, $y = (ab)_c^3$ and $\Delta(x) = \{abc, (abc)^2\}$, $\Delta(y) = \{cab, (cab)^2\}$. Therefore,

$$\deg(x) = \deg(y) = 3, \quad |\Delta(x)| = |\Delta(y)| = 2,$$

and hence Theorem 3.5 implies $X_x^* \cong X_y^*$. But $x \notin P(X)$ and $y \in P(X)$, thus words x and y do not satisfy conditions of Theorem 4.2, i.e. $X_x^+ \ncong X_y^+$.

2) Now let $x = ababa$ and $y = 1011010110 \in \{0, 1\}^+$. We have

$$x = (a)_b^3, \quad y = (10110)^2 = (10)_{110101}^2$$

and

$$\Delta(x) = \{ba, (ba)^2\}, \quad \Delta(y) = \{10110, (110101)10\}.$$

Hence

$$\deg(x) = 3 = \deg(y) + 1, \quad |\Delta(x)| = |\Delta(y)| = 2,$$

and $x \in P(X)$, $y \notin P(\{0, 1\})$. According to Theorem 4.2, we have $X_x^+ \cong \{0, 1\}_y^+$ and Theorem 3.5 implies that $X_x^* \ncong \{0, 1\}_y^*$.

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